Geofest Design Document

1. Introduction

Geofest uses stress-displacement finite elements to model stress and strain due to elastic static response to an earthquake event in the region of the slipping fault, the time-dependent viscoelastic relaxation, and the net effects from a series of earthquakes. The physical domain may be two- or fully three-dimensional and may contain heterogeneous rheology and an arbitrary network of faults. The software is intended to simulate viscoelastic stress and flow in a realistic model of the earth's crust and upper mantle in a complex region such as the Los Angeles Basin.

2. Mathematical Equations for the Visco-Elastic Mechanics Problem

We describe the quasi-static mathematical equations for visco-elastic materials, which is the assumed material type of the solid earth being modeled. In the following, σ and ε denote second-order stress tensors for stress and strain fields, respectively, and u is the displacement field. The summation convention is used for repeated indices; a comma is used to denote a partial derivative with respect to a spatial dimension in a Cartesian coordinate system. In R^3, for example, we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \sigma_{ij,j} = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3}$$

The considered equations include

$$\sigma_{ij,j} + f_i = 0, \tag{1.1}$$

the equilibrium equation, where f_i is the given body force,

$$\frac{\partial \sigma_{ij}}{\partial t} = c_{ijkl}\left(\frac{\partial \varepsilon_{kl}}{\partial t} - \frac{\partial \varepsilon_{kl}^{vp}}{\partial t}\right), \tag{1.2}$$

the constitutive equation, where c_{ijkl} are material-specific constants, and

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.3}$$

$$\frac{\partial \varepsilon_{ij}^{vp}}{\partial t} = \beta_{ij}(\sigma_{ij}), \tag{1.4}$$

where ε^{vp} is the viscoplastic strain, and β_{ij} are viscoplastic strain rates which are given functions of the stress field. The problem to be solved is formulated as an initial-boundary-value problem in a domain $\Omega \subset R^n$, where $n = 2$ or 3. We want to find a displacement field $u(x,t)$ and a stress tensor field $\sigma_{ij}(x,t)$ which satisfy equations (1.1) to (1.4) for all $x \in \Omega$ and $t \in [0,T]$, $T > 0$, such that

$$
\begin{aligned}
&u_i(x,t) = u_0(x), \quad x \in \Omega \\
&\sigma_{ij}(x,0) = \sigma_{0ij}(x), \quad x \in \Omega \\
&u_i(x,t) = g_i(x,t), \quad x \in \partial\Omega_1,\ t \in [0,T] \\
&\sigma_{ij} n_j = h_i(x,t), \quad x \in \partial\Omega_2,\ t \in [0,T]
\end{aligned}
\tag{1.5}
$$

where u_0 and σ_0 are the initial displacement and stress fields, respectively, $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$ is the domain boundary, n is an outward normal vector to $\partial\Omega_2$, and $g_i(x,t)$ and $h_i(x,t)$ are prescribed boundary displacement and tractions, respectively.

For isotropic (Newtonian) material, the material constants in (1.2) can be expressed as

$$
c_{ijkl} = \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda(x)\delta_{ij}\delta_{kl} \tag{1.6}
$$

where λ and μ are known as *Lame parameters,* which are related to Yong's modulus E and Poisson's ratio ν by

$$
\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.
$$

3. Finite Element Formulation

In a finite element approximate solution to problem (1.1) - (1.5), we seek an approximate displacement field $u_i(x,t) \in S$, where S is a finite-dimensional trial solution space with each u_i in S satisfying $u_i = g_i$ (the essential boundary condition) on $\partial\Omega_1$. We also define a finite-dimensional variation space V_i with each $w_i \in V$ satisfying $w_i = 0$ on $\partial\Omega_2$. u_i must satisfy the "weak form" of the problem (1.1) - (1.5), given below:

$$
\boxed{
\begin{aligned}
&\text{Find } u_i \in S_i \text{ such that for all } w_i \in V_i, \\
&\int_\Omega w_{(i,j)} \sigma_{ij} d\Omega = \int_\Omega w_i f_i d\Omega + \sum_{i=1}^{n} \Big(\int_{\partial\Omega_2} w_i h_i d\Omega \Big)
\end{aligned}
}
\tag{2.1}
$$

where $w_{(i,j)} = w_{i,j} + w_{j,i}$, σ_{ij} is related to u_i through (1.2) and (1.3), and n is the spatial dimension. w_i is sometimes referred to as *virtual displacements* in solid mechanics. Under some smoothness assumptions on the involved variables, it can be shown that a solution to (2.1) is a solution to (1.1) - (1.5) and vice versa.

To find a numerical solution to the finite element problem (2.1), all the variables and the integral equation in (2.1) are discretized on a *finite element mesh*. In the Geofest program implementation, the discrete displacement field u^h is defined at nodal points of the mesh, and stress field σ^h and strain field ε^h are defined at the center of a mesh cell (an element).

Using the definition of (1.6), and using a certain mapping of the indices of i, j, k, l to indices I, J [2], it can be shown that

$$\sigma = D\varepsilon(u) \tag{2.2}$$

where, in R^2,

$$D = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \lambda+2\mu \end{bmatrix}, \qquad \varepsilon(u) = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{Bmatrix}.$$

Now define

$$a(w,u) = \int_{\Omega} \varepsilon(w)^T D\varepsilon(u) d\Omega .$$

Let

$$u^h = v^h + g^h ,$$

where $u^h = \left\{u_1^h, \cdots, u_n^h\right\}^{\mathrm{T}}$, $v^h = \left\{v_1^h, \cdots, v_n^h\right\}^{\mathrm{T}}$ and $g^h = \left\{g_1^h, \cdots, g_n^h\right\}^{\mathrm{T}}$ are vectors in R^n, $v_i^h \in V_i$ vanishes on $\partial\Omega_2$, g_i^h satisfies the boundary conditions on $\partial\Omega_1$, so $u_i = g_i$ on $\partial\Omega_1$. In particular, let

A = total nodes in the mesh.

η_{eb} = set of nodes on which $u_i = g_i$,

and

$$v_i^h = \sum_{a\in(A-\eta_{eb})} N_a d_a, \quad g_i = \sum_{a\in\eta_{eb}} N_a g_a$$

where N_a is the "shape function" associated with node a; N_a takes unit value at node a and vanishes on neighboring nodes of a; d_a is the displacement value at node a which is an unknown to be computed. Let e_i be a basis vector in R^n with its i-th component equal to one and other components equal to zero. We have

$$v^h = v_i^h e_i, \quad g^h = g_i^h e_i.$$

Also let

$$w^h = w_i^h e_i, \quad w_i^h = \sum_{a\in(A-\eta_{eb})} N_a c_a$$

where c_a are arbitrary constants.

Substituting the previous definitions into (2.1), we get a matrix equation for the displacement vector d

$$Kd = F(= F_1 + F_2) \tag{2.3}$$

where $K = [k_{pq}] \in R^{m\times m}$ is the so-called stiffness matrix. K is symmetric and positive definite, and

$$m = \sum_{a\in A} n_a^{dof}$$

where n_a^{dof} is the degree of freedom at node a. An entry of matrix K, k_{pq}, has the form

$$k_{pq} = a(N_a e_i, N_b e_j) = \int_\Omega \varepsilon(N_a e_i)^T D\varepsilon(N_a e_i) d\Omega = e_i \int_\Omega B_a D B_b d\Omega e_j$$

where global equation numbers p, q and global node numbers a, b are related through a certain defined mapping. In R^2

$$B_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix}.$$

F_1 and F_2 on the right-hand side of (2.3) are known vectors in R^m. F_1 includes the contributions from the body force and boundary condition terms. And $F_2 = \int_\Omega B_a^T D\varepsilon^{vp} d\Omega$ is the contribution from the viscoplastic strain.

4. An Implicit Time-Stepping Scheme (Hughes & Taylor)

A time-stepping scheme is needed to compute a visco-elastic finite element solution of displacement and stress fields at discrete time points over a given time period. Both explicit and implicit time-stepping schemes can be formulated. The Geofest program adopted an implicit scheme because of its unconditional numerical stability with respect to time step sizes. The entire solution process consists of an initial solve of a pure elastic problem for which the viscoplastic strain rate is set to zero. The pure elastic solution provides an initial stress field, which is then relaxed over a time period in a visco-elastic solve for which an implicit stepping scheme is used. This algorithm used by Geofest is described in the following steps:

1. Initialize, set $n = 0$
   a. Form K_0 and f_0
   b. Solve $Ku_0 = f_0$
   c. $\sigma_0 = DBu_0$
2. Form step stiffness matrix and right-hand side
$$K_{n+1} = \int_\Omega B^T (S + \alpha \Delta t \beta_n^{'})^{-1} B d\Omega$$
$$F_{n+1} = \int_\Omega B^T (S + \alpha \Delta t \beta_n^{'})^{-1} (\Delta t \beta_n) d\Omega + f_{n+1}$$
   where $S = D^{-1}$, $0 < \alpha < 1$.
3. Solve $K_{n+1} \delta u_{n+1} = F_{n+1}$
4. Stress increment: $\delta\sigma_{n+1} = (S + \alpha \Delta t \beta_n^{'})^{-1} (B \delta u_{n+1} - \Delta t \beta_n)$
5. Update displacement and stress fields
$$u_{n+1} = u_n + \delta u_{n+1}$$
$$\sigma_{n+1} = \sigma_n + \delta\sigma_{n+1}$$

6. If (last_time_step)
    stop
   Else
    set $n = n+1$,
    go back to 2.

In the above scheme, the viscoplastic strain rate, $\beta(\sigma)$, and its Jacobian matrix, $\beta^{'}(\sigma)$, need to be specified. In R^2, they are

$$\beta(\sigma) = \frac{\kappa}{4\eta}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \beta^{'}(\sigma) = \frac{\kappa}{4\eta}\begin{bmatrix} a & -a & d\sigma_{xy} \\ -a & a & -d\sigma_{xy} \\ d\sigma_{xy} & -d\sigma_{xy} & 4b \end{bmatrix}$$

where

$$\kappa = \sqrt{(\frac{\sigma_{xx} - \sigma_{xy}}{2})^2 + \sigma_{xy}^2}$$

$$a = 1 + (\frac{\sigma_{xx} - \sigma_{xy}}{2\kappa})^2, \quad b = 1 + (\frac{\sigma_{xy}}{\kappa})^2, \quad d = (\frac{\sigma_{xx} - \sigma_{yy}}{\kappa^2})$$

5. Fault Specification and Split Node Implementation

Fault conditions can be specified either as fault elements or as split nodes. With fault elements, one can specify fault properties such as failure criterion. With split nodes, one can represent the rate of displacement of a fault surface by assigning the direction and amount of slip for each node on the fault surface. Typically, a split node has different slip rates assigned to it on each side of the fault surface, which introduces a discontinuity in the displacement field to simulate real fault slip. This idea can be illustrated by a simple one-dimensional example with two elements, as shown in Figure 1.

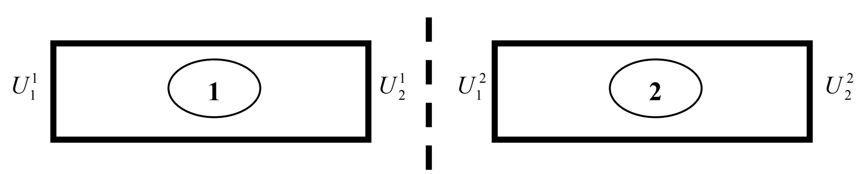

Figure 1

It is assumed that elements 1 and 2 are located adjacent to the opposite sides of the fault surface represented by a dash line between the two elements, and U is the displacement field. Away from the fault, displacement field has a single value defined at each node of the 1-D finite element mesh, such as U_1^1 on the left node of element 1 and U_2^2 on the right node of element 2. The node between the two elements is considered a split node since it lands on the fault. The displacement field has different values at the split node, which are U_2^1 on the side of elements *A* and U_1^2 on the side of element *B*. Specifically we can write

$$U_2^1 = \overline{U_2^1} + \Delta U_2^1, \quad U_1^2 = \overline{U_1^2} + \Delta U_2^1$$

where $\overline{U_2^1} = \overline{U_1^2}$ is the mean value of displacement at the split node, and $\Delta U_2^1 = -\Delta U_1^2$ is the "splitting" part of displacement that has opposite signs on two sides of the fault line. In a finite element implementation, the contribution from the splitting displacements can be formulated as an additional forcing term. This fact can also be shown using the two-element example. The local stiffness matrix for element 1 can be written as

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} \begin{bmatrix} U_1^1 \\ \overline{U_2^1} + \Delta U_2^1 \end{bmatrix} = \begin{bmatrix} F_1^1 \\ F_2^1 \end{bmatrix},$$

which relates local displacements to local force terms. By moving the known quantities of the above equation to the right-hand side, we have

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 \\ K_{21}^1 & K_{22}^1 \end{bmatrix} \begin{bmatrix} U_1^1 \\ \overline{U_2^1} \end{bmatrix} = \begin{bmatrix} F_1^1 - K_{12}^1 \Delta U_2^1 \\ F_2^1 - K_{22}^1 \Delta U_2^1 \end{bmatrix} \tag{3.1}$$

.

Similarly for element 2, we have

$$\begin{bmatrix} K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{bmatrix} \overline{U_1^2} \\ U_2^2 \end{bmatrix} = \begin{bmatrix} F_1^2 - K_{12}^2 \Delta U_1^2 \\ F_2^2 - K_{22}^2 \Delta U_1^2 \end{bmatrix}. \tag{3.2}$$

"Assembling" the local stiffness matrix equations into a global stiffness matrix equation, we get

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 \\ K_{21}^1 & K_{22}^1 + K_{12}^2 & K_{12}^2 \\ 0 & K_{21}^2 & K_{22}^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_1 - K_{12}^1 \Delta U_2^1 \\ F_2 - K_{22}^1 \Delta U_2^1 - K_{11}^2 \Delta U_1^2 \\ F_3 - K_{21}^2 \Delta U_1^2 \end{bmatrix} \tag{3.3}$$

where U_i's are global displacements, which are related to the node local displacements by

$$U_1 = U_1^1, U_2 = U_2^1 = U_1^2, U_3 = U_2^2.$$

The global force terms F_i are related to the local ones by

$$F_1 = F_1^1, F_2 = F_2^1 + F_{1,}^2\ F_3 = F_2^2.$$

Equations (3.1)-(3.3) show that the effect of the slips on the split nodes is equivalent to adding those additional terms on the right-hand side of the finite element matrix equations.

Stress and displacement at each time are the accumulations of incremental stresses and displacements for past time steps. When a slip event occurs, the incremental displacements are found by applying the split nodes adjustments to the right hand side of the stiffness equation. After the incremental displacement is obtained, the incremental stress is found by including the split node contribution to the stress for that time step. In this way the displacement and stress effects of a slip event are correctly carried forward into future time steps, without any need for additional storage for the slip history of the fault.

Reference:

[1] Thomas J. R. Hughes and Robert Taylor, "Unconditionally Stable Algorithms For Quasi-Static Elasto-Plastic Finite Element Analysis." Computers & Structures, Vol. 8, pp. 169-173, 1978

[2] Thomas J. R. Hughes, "The Finite Element Method: Linear Static and Dynamic Finite Element Analysis." Dover, Publication, INC., Mineola, New York, 2000

[3] H. J. Melosh and Raefsky, "A Simple and Efficient Method for Introducing Faults into Finite Element Computations." Bulletin of the Seismological Society of America, Vol. 71, No. 5, October, 1981